Dexter semi-lattices and Hochschild polytopes

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Many partial orders on Catalan objects

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→ (A) noncrossing partitions for refinement (Kreweras),
→ (B) binary trees and rotation moves (Tamari),
→ (C) binary trees under left-arm rotation order (Pallo),
→ (D) Dyck paths for inclusion,
→ (E) Dyck paths and Tamari sliding moves (equivalent to (B)),
→ (F) Dyck paths and total sliding moves,
and still others by restriction from the symmetric groups.
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Today, introduce yet another one:
→ Dyck paths and dexter sliding moves.
Informal motivation (diagonals of associahedra $K_n$)

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Partial order: $(S, T) \leq (S', T')$ iff $S \leq S'$ and $T \leq T'$. 
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Note the natural (visual) partition into cells.
Informal motivation (Catalan-many cells)

In this picture, unique cell containing the top \( \simeq \) Tamari lattice

The unique top cell
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**Claim**: every vertex of this cell is the **top element** of a cell!
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\(\text{Claim: every vertex of this cell is the } \text{top element} \text{ of a cell!}\)
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the unique top cell and the cells below its vertices

**Claim**: every vertex of this cell is the *top element* of a cell!

This gives Catalan-many cells (among many more cells A0139).

$\rightarrow$ induced partial order on the *bottom elements* of these cells
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Direct combinatorial description of Dexter posets

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For the Tamari lattice: slide any sub-Dyck path (after a descent) by one NW step.

For the Dexter poset: slide any sub-Dyck path (after a descent) not followed by a descent by one or several NW steps.
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For the **Dexter poset**: slide any sub-Dyck path (after a descent) not followed by a descent by one or several NW steps.

→ Every dexter sliding move is like a sequence of Tamari sliding moves, so something like a **shortcut** in the Tamari lattice.
More examples of sliding moves

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one single dexter sliding move. This would be two consecutive moves in the Tamari lattice.
Comparison between Tamari and Dexter

Dexter on the left and Tamari on the right
Comparison between Tamari and Dexter

Dexter on the left and Tamari on the right

Tamari has strictly more relations than Dexter.
Picture of the next full Dexter poset
Can you see the hidden pentagon?
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→ unique minimal element: 

Theorem

The dexter poset $D_n$ is a meet-semilattice.

→ every pair of elements has a unique common lower bound.

(not the same as in the Tamari lattice)

Proof

$≃$ closing a zipper from left to right.

Corollary

Every interval in $D_n$ is a lattice.
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*Every interval in $\mathcal{D}_n$ is a lattice.*
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The number of intervals in the poset $\mathcal{D}_n$ is 1 for $n = 0$ and

$$3 \frac{2^{n-1}(2n)!}{n!(n+2)!} \quad \text{for} \quad n \geq 1.$$
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→ sequence A257 : 1, 1, 3, 12, 56, 288, 1584, 9152, . . . (Tutte)
→ (A) numbers of rooted bicubic planar maps on \( 2n \) vertices
→ (B) numbers of rooted Eulerian planar maps with \( n \) edges
→ (C) numbers of modern intervals in the Tamari lattices
→ (D) numbers of new intervals in the Tamari lattices
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Bijection between (A) and (B) is classical (Tutte).
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About the proof (bijectons and formulas)

The proof uses a recursive bijective description of all the intervals. The good **catalytic parameter** turns out to be the number of **blocks** (returns to zero) of the minimum of the interval. → functional equation

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f = 1 + st + st(f - 1) \left( 1 + \frac{sf - f|_{s=1}}{s - 1} \right) + tf|_{s=1}.
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Using the general method of Bousquet-Mélou and Jehanne, one obtains an algebraic equation

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16g^2 t^2 - g(8t^2 + 12t - 1) + t^2 + 11t - 1
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→ the known algebraic equation for the sequence A257
A very special interval

General picture of the dexter order looks like the union of some polyhedra. Show What are these polyhedra?
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Exercise: they are indeed comparable in the dexter order!
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Call $F_n$ the set of elements in this interval (with $n$ little peaks). This is a lattice (because all intervals are).
The Hochschild polytope

In algebraic topology, Saneblidze has introduced a family of polytopes, called the **Hochschild polytopes** used to make combinatorial cellular models of free loops spaces.
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→ definition (2) using explicit (recursive) “cubical coordinates”
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The interval $F_n \simeq$ the Hochschild polytope of dimension $n$. The number of elements of $F_n$ is $2^{n-2}(n+3)$.

namely 2, 5, 12, 28, 64, 144, 320, . . . .
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The $h$-vector should be nice too : $(x + 1)^{n-2}(x^2 + (n + 1)x + 1)$
Not graded, hence not distributive. Maybe a trim lattice?
→ Counting **edges in the Hasse diagrams** of $\mathcal{D}_n$
With no colors: same $h$-vector as the associahedra (Narayana numbers)
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→ the **diameter** of $F_n$ seems to be $n$
(for those among you who like computing diameters)
Bonus tracks (conjectures, remarks, speculations)

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intriguing appearance of A7852 **Antichains in rooted plane trees**
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→ the dexter lattice $D_n$ is not derived equivalent to the Tamari lattice
Questions ?