Une notion de récurrence dans le modèle du tas de sable sur le réseau carré

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joint work with Yvan Le Borgne

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Sandpile Model [Bak, Tang, Wiesenfeld 87]

Configuration: $\eta : V \mapsto \mathbb{N}$

$\nu \in V$ is unstable for $\eta$ if $\eta(\nu) \geq \deg(\nu)$, it is stable otherwise.
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$\nu \in V$ is unstable for $\eta$ if $\eta(\nu) \geq \deg(\nu)$, it is stable otherwise.

Toppling $u: \eta \mapsto \eta + \Delta^{(u)}$ If $u$ is unstable, the toppling is legal. It is forced otherwise.
Configuration: $\eta : V \mapsto \mathbb{N}$

$v \in V$ is unstable for $\eta$ if $\eta(v) \geq \deg(v)$, it is stable otherwise.

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$v \in V$ is *unstable* for $\eta$ if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.

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Sandpile Model [Bak, Tang, Wiesenfeld 87]

Configuration: \( \eta : V \mapsto \mathbb{N} \)

\( v \in V \) is *unstable* for \( \eta \) if \( \eta(v) \geq \deg(v) \), it is *stable* otherwise.

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A vertex $v \in V$ is unstable for $\eta$ if $\eta(v) \geq \deg(v)$, it is stable otherwise.

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The order of toppling does not change the result: $\eta \rightarrow \eta + \sum_{v \in V} a_v \Delta(v)$. 

H. Derycke, Y. Le Borgne (LaBRI)  
Square lattice, Sandpile and Recurrence  
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The order of toppling does not change the result: $\eta \rightarrow \eta + \sum_{\nu \in V} a_{\nu} \Delta^{(\nu)}$. 
Stabilisation : while a vertex is unstable, topple it.
The sink

How to stabilize (even with a large number of grains)?

\[
\begin{array}{cc}
2 & 2 \\
3 & 3 \\
2 & 2 \\
\end{array}
\]
The sink

How to stabilize (even with a large number of grains)?

We distinguish a vertex as the sink that won’t topple.
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The sink guarantees that the stabilisation of any configuration $\eta$ terminates and we note the result $\mathsf{stab}(\eta)$.
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![Stabilisation](image)

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**Markov Chain**

- States: stable configurations on $G$
- Transition: Add a particle from the sink to a vertex chosen uniformly and stabilize

Recurrent states are in the same connected component.
The sink

How to stabilize (even with a large number of grains)?

\[
\begin{array}{c|c|c|c}
2 & 2 \\
3 & 3 \\
2 & 2 \\
\end{array}
\]
\[
\begin{array}{c|c|c|c}
0 & 1 \\
2 & 2 \\
9 & 1 \\
\end{array}
\]
\[
\begin{array}{c|c|c|c}
1 & 1 \\
2 & 2 \\
8 & 1 \\
\end{array}
\]

We distinguish a vertex as the sink that won’t topple.

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Test of recurrence

Dhar operator
Topple the sink (forced), then stabilize: $dhar(\eta) := \text{stab}(\eta + \Delta^{(s)})$
Test of recurrence

Dhar operator

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\[
\begin{array}{c|c|c|c|c|c|}
0 & 1 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 1 & 1 \\
s & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]
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Dhar Criterion

A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.
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![Dhar operator example](image)

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![Diagram of sandpile model and Dhar operator application]

**Dhar Criterion**

A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.

![Diagram illustrating the Dhar criterion application]
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Topple the sink (forced), then stabilize: \( \text{dhar}(\eta) := \text{stab}(\eta + \Delta^{(s)}) \)

\[
\begin{array}{c|c}
0 & 1 \\
2 & 2 \\
s & 1 \\
\hline
3 & 2 \\
& s \\
\hline
0 & 4 \\
& s \\
\hline
1 & 2 \\
& s \\
\hline
2 & 0 \\
& s \\
\hline
0 & 1 \\
& s
\end{array}
\]

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\[
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0 & 1 \\
2 & 0 \\
s & 1 \\
\hline
3 & 0 \\
& s \\
\hline
0 & 2 \\
& s \\
\hline
1 & 0 \\
& s
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\[
\begin{array}{cccc}
0 & 1 & 0 & 1 \\
2 & 2 & 3 & 2 \\
s & 1 & s & 2 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
1 & 1 & 0 & 4 \\
1 & 2 & s & 0 \\
1 & 1 & s & 1 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
2 & 0 & 1 & 2 \\
2 & 1 & s & 1 \\
0 & 1 & s & 1 \\
\end{array}
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Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph $G$ and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce’s path for sorted recurrences on $K_n$)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)

Mark edges incident to the sink as pending edges.
While there is a pending edge
Get the closest pending edge to the sink
Process the grain(s) on the edge
If a vertex become unstable, topple it and mark its untreated incident edges as pending edges.

Edge-vertex traversal: $s$, $v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$
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Diagram:

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Edge-vertex traversal: \( s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7, v_5, e_8, v_6, e_9 \)
Counting grains

In Dhar criterion, each edge captures the last grain that crosses it. For any recurrent configuration \( \eta \) on \( G = (V \cup \{s\}, E) \),

\[
\text{level}(\eta) = \left( \sum_{v \in V} \eta(v) \right) + \deg(s) - |E|.
\]

Let \( R_G(y) = \sum_{\eta \in \text{Rec}(G,s)} y^{\text{level}(\eta)} \)

**Theorem (López 97)**

For any graph \( G = (V \cup \{s\}, E) \),

\[
R_G(y) = \text{Tutte}_G(1, y).
\]

where \( \text{Tutte}_G(1, y) = \sum_{T \in \Sigma_G} y^{\text{ext}(T)} \) counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.
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where $\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.
Tracking external activity while changing order on edges

With $e_1 <_E e_2 <_E \cdots <_E e_{|E|}$ an order on the edges of $E$, an external edge is *active* if it is maximal for $<_E$ in its fundamental cycle.

**Proposition**

$$\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}_{<_E}(T)}$$

does not depend on $<_E$.
Tracking external activity while changing order on edges

With $e_1 <_E e_2 <_E \cdots <_E e_{|E|}$ an order on the edges of $E$, an external edge is active if it is maximal for $<_E$ in its fundamental cycle.

{\text{Proposition}}

$$\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}_E(T)}$$

does not depend on $<_E$

\{e_i, e_j\} is a critical pair if

- $e_i$ is external
- $e_j$ is on $e_i$ fundamental cycle
- $e_i$ and $e_j$ are maximal on $e_i$ fundamental cycle
Tracking external activity while changing order on edges

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\{$e_i, e_j$\} is a critical pair if

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- $e_i$ and $e_j$ are maximal on $e_i$ fundamental cycle

Let $\tau_i$ exchanging $e_i$ and $e_{i+1}$ in $<_E$.

$$\Phi_i(T) = \begin{cases} T \Delta \{e_i, e_{i+1}\} & \text{if } \{e_i, e_{i+1}\} \text{ is a critical pair of } T \\ T & \text{otherwise} \end{cases}$$

Lemma: for all $T$ $\text{ext}_{<_E}(T) = \text{ext}_{\tau_i(<_E)}(\Phi_i(T))$. 

H. Derycke, Y. Le Borgne (LaBRI) Square lattice, Sandpile and Recurrence JCB 2019 8 / 29
Tutte Polynomial

Let a graph \( G = (V, E) \) and \(<_E\) an order on the edges of \( E\).

\[
\text{Tutte}_G(x, y) = \sum_{T \in \Sigma(G)} x^\text{int}(T) y^\text{ext}(T)
\]

Active external edge: maximal in its fundamental cycle.
Active internal edge: maximal in its co-cycle.

For \( G = K_4 \), \( \text{Tutte}_G(x, y) = x^3 + y^3 + 3x^2 + 4xy + 3y^2 + 2x + 2y \) and \( T \) weights \( xy \).

When \( G \) is planar, \( \text{Tutte}_G(x, y) = \text{Tutte}_{G^*}(y, x) \). Then if planar and self-dual, \( \text{Tutte}_G(x, y) = \text{Tutte}_G(y, x) \).
Finite graphs

- Stable configurations

- Dhar Criterion

- Bijection between recurrent and spanning trees

- Tutte polynomial

- Invariant by edge exchange

- Symmetric for self-dual planar graphs
## Checkpoint

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Finite graphs

- Stable configurations
- Dhar Criterion
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- Invariant by edge exchange
- Symmetric for self-dual planar graphs

Square lattice (biperiodicity)

- Biperiodic stable configurations
- Weak Dhar Criterion (projective sink)
- Bijection recurrent and some spanning forests of the torus
- Restriction of Tutte polynomial
- Distribution of external activity invariant by rotation of projective sink

Symmetric for self-dual planar graphs
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<td>▶ Symmetric joint distribution of external/internal activities <em>changing</em> by rotation</td>
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Some definition of recurrence for $\mathbb{Z}^2$

From wired uniform spanning forest [Gamlin, Jarai] with an anchor burning bijection.

Local description in probability [Priezzhev, Ruelle]

Sandpile identity: $\lim_{n \to \infty} \operatorname{dhar}^n(0^{\mathbb{Z}^2})$? [Paoletti, Caracciollo, Sportiello, Levine, Pegden, Smart...]

Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]

Convergence in terms of density [Pegden, Smart 12]
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Convergence in terms of density [Pegden, Smart 12]

Source: W. Pegden, $n = 2^{13}$
Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]

Convergence in terms of density [Pegden, Smart 12]

Source: W. Pegden, $n = 2^{14}$
Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]
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Convergence in terms of density [Pegden, Smart 12]
Source: W. Pegden, $n = 2^{30}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]
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Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent?
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Pattern in periodic zones are invariant when toppling the sink ⇒ recurrent?
Heuristic: locally, toppling the sink behave as the toppling of an half-plane.
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbb{Q}^2 \ (\neq (0, 0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

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Direction $\vec{s}$ du puits
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\[ \xrightarrow{+1} \]

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\[+1\]

Direction \( \vec{s} \) du puits
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

Direction $\vec{s}$ du puits
Definition (Weak Dhar Criterion [D., Le Borgne 2018])
A stable configuration is recurrent for a direction $\vec{s} \in \mathbb{Q}^2$ (≠ (0, 0)) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

\[ +1 \]

\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
2 & 2 & 1 & 3 & 2 & 2 & 2 \\
\hline
2 & 2 & 1 & 3 & 2 & 2 & 2 \\
\hline
2 & 2 & 1 & 3 & 2 & 2 & 2 \\
\hline
2 & 2 & 1 & 3 & 2 & 2 & 2 \\
\hline
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\hline
\end{array}

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![Diagram of sandpile model](image-url)
Demo
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.

\[\begin{array}{ccccccccccc}
3 & 3 & 3 & 3 & 3 & 0 & 3 & 3 & 3 & 3 & 0 & 3 \\
1 & 3 & 0 & 3 & 3 & 1 & 3 & 0 & 3 & 3 & 1 \\
1 & 3 & 3 & 1 & 3 & 1 & 3 & 3 & 1 & 3 & 1 \\
1 & 3 & 0 & 3 & 2 & 1 & 3 & 0 & 3 & 2 & 1 \\
\end{array}\]

sweep
line

Periodicity along the orthogonal of the sink
Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.
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*The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.*

![Diagram of a sweep line and periodicity along the orthogonal of the sink.](image)
Theorem (D., Le Borgne 2018)

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![Diagram of sandpile with sweep line and periodicity along the orthogonal of the sink]
**Theorem (D., Le Borgne 2018)**

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.

<table>
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**Frozen**

**Working zone**

```
1 3 0 3 2 1 3 0 3 2 1
1 1 1 2 3 1 1 1 2 3 1
```

Pericodiity along the orthogonal of the sink

Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane

H. Derycke, Y. Le Borgne (LaBRI) Square lattice, Sandpile and Recurrence JCB 2019 16 / 29
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Periodicity along the orthogonal of the sink
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The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\mathbf{s}$.

- Periodicity along the orthogonal of the sink
- Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane
sweep line
Spanning forests of the torus rooted on non-contractible cycles with slope $(4, -3)$.

Periodic spanning forest rooted on the half-plane spanning forest of the torus with slope $(1, 0)$ incompatible with the vertical direction.

Theorem [D., Le Borgne 2018] Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $F_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.
Spanning forests of the torus rooted on non contractible cycles with slope \((4, -3)\).

Periodic spanning forest rooted on the half-plane.

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Biperiodic spanning forest with infinite paths directed towards the sink.

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H. Derycke, Y. Le Borgne (LaBRI) Square lattice, Sandpile and Recurrence JCB 2019 18 / 29
Spanning forests of the torus rooted on non-contractible cycles with slope $\left(\frac{4}{3}, -\frac{3}{3}\right)$

Biperiodic spanning forest with infinite paths directed towards the sink
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Spanning forest of the torus with slope \((1,0)\) incompatible with the vertical direction

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Recurrent configurations of period \(W \times H\) defined by weak Dhar criterion with projective sink in direction \(\vec{s}\) are in bijections with admissible forests of \(\mathcal{F}_{W \times H, \vec{s}}\), hence excluding those of slope orthogonal to \(\vec{s}\).
Determinantal formula [Kenyon 17] for non contractible cycle rooted spanning forests (NCRSFs)

Refinement with the infinite path’s slope

\[
\begin{array}{c|cccc}
  k \cdot j & k \cdot i \\
  \hline
  0 & 31300528 & 541732 & 1528 & 1 \\
  1 & 31300528 & 5427200 & 31232 & 4 \\
  2 & 541732 & 31232 & 6 \\
  3 & 1528 & 4 \\
  4 & 1 \\
\end{array}
\]

**Table:** Number of NCRSFs with \( k \) cycles of slope \((i, j)\) on the torus \( T_{4,4} \)

Computation for \( W, H \leq 9 \)
Inverse function

Placing the grains on the edges. ⚪ ⬤
Inverse function

Placing the grains on the edges. ● ○

- Orientation towards the sink
Inverse function

Placing the grains on the edges.

- Orientation towards the sink
- Internal: 1 grain • to the father
Inverse function

Placing the grains on the edges. □ ○

- Orientation towards the sink
- Internal: 1 grain ● to the father
- External: ● depends on the position of the maximal edge on the fundamental cycle
Inverse function

Placing the grains on the edges. ⬜️ ○

- Orientation towards the sink
- Internal: 1 grain ⬜️ to the father
- External: ⬜️ depends on the position of the maximal edge on the fundamental cycle

Red ray

Blue ray
Inverse function

Placing the grains on the edges. ● ○

- Orientation towards the sink
- Internal: 1 grain ● to the father
- External: ● depends on the position of the maximal edge on the fundamental cycle
Inverse function

Placing the grains on the edges. ▶️ ○

- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

\[ s \]

\[ e_{\text{max}} \]

\[ s \rightarrow e \]
Inverse function

Placing the grains on the edges. ● ○

- Orientation towards the sink
- Internal: 1 grain ● to the father
- External: ● depends on the position of the maximal edge on the fundamental cycle

![Diagram of a sandpile model with grains placed on edges and arrows indicating the direction of grain movement.](image)
Inverse function

Placing the grains on the edges. 

- Orientation towards the sink
- Internal: 1 grain \( \bullet \) to the father
- External: \( \bullet \) depends on the position of the maximal edge on the fundamental cycle

\[ e_{\text{max}} \]

\[ s \]
Inverse function

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- Orientation towards the sink
- Internal: 1 grain ● to the father
- External: ● depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active

\( e_{\text{max}} \)

\( s \)
Inverse function

Placing the grains on the edges. • ○

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- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active
Inverse function

Placing the grains on the edges. ⬤ ○

- Orientation towards the sink
- Internal: 1 grain ⬤ to the father
- External: ⬤ depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active
Inverse function

Placing the grains on the edges. • ○

- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active

Cycles are directed such that they are globally decreasing. Periodicity ⇒ Maximal edge at finite distance
Inverse function

Placing the grains on the edges. ⬜️️️

- Orientation towards the sink
- Internal: 1 grain ⬜️ to the father
- External: ⬜️ depends on the position of the maximal edge on the fundamental cycle
- External: ⬜️ on the other endpoint if active

Cycles are directed such that they are globally decreasing.
Periodicity ⇒ Maximal edge at finite distance
Restricted Tutte Polynomial

\[ \mathcal{T}_{W \times H, s}(x, y) = \sum_{T \in \mathcal{F}_{W \times H}} x^{\text{int}_{W \times H}(T)} y^{\text{ext}_{W \times H}(T)} \]

\( e <_s f \) if \( e \) is closer to the sink than \( f \).

Restrictions
- On NCRSF: \( \mathcal{F}_{W \times H} \).
- On the activity: on the rectangular fundamental domain \( W \times H \) consider exactly 2WH edges.
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External activity on $F_{3,1}$:

\[
\begin{array}{ccc}
    & \uparrow & \leftarrow \\
    s & 3 & 0 & 0 \\
    \uparrow & 2 & 1 & 1 \\
    \leftarrow & 1 & 0 & 2 \\
\end{array}
\]

\[
T_{3 \times 1,(0,1)}(1, y) = y^3 + 3y^2 + 6y + 7
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T_{3 \times 1,(-1,0)}(1, y) = y^3 + 3y^2 + 6y + 7
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Theorem (D., Le Borgne 2018)

For any directions $s, s'$, $\mathcal{T}_{W \times H, s}(1, y) = \mathcal{T}_{W \times H, s'}(1, y)$. 

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Since $\mathbb{Z}^2$ is self-dual, we have:

\[
\mathcal{T}_{3 \times 1, (0,1)}(x, y) = x^3y^3 + 3xy^2 + 3x^2y + 3x + 3y + 4
\]

\[
\mathcal{T}_{3 \times 1, (-1,0)}(x, y) = x^3y^3 + 3x^2 + 3y^2 + 3xy + 3x + 3y + 1
\]
External activity

Direction of the sink

For each external edge $e$, there is an activity sector $[\theta_e, \theta'_e)$.
For any sector excluding all $(\theta_e)$ and $(\theta'_e)$, the external activity is invariant.
External activity

Direction of the sink

- Convex hulls of fundamental cycles.
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- Convex hulls of fundamental cycles.
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Critical pair exchange: Rotation step

\[ s = (1, 0) \]
Critical pair exchange: Rotation step

\[ s = (1, 1) \]
Critical pair exchange: Rotation step

\[ s = (1, 1) \]
Critical pair exchange: Rotation step

$s = (0, 1)$
Critical pair exchange: Rotation step

\[ s = (0, 1) \]
Critical pair exchange: changing forest slope
<table>
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<tr>
<th>Finite graphs</th>
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<td>▶ Biperiodic stable configurations</td>
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<td>▶ Dhar Criterion</td>
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<td>▶ Symmetric for self-dual planar graphs</td>
<td>▶ Symmetric joint distribution of external/internal activities changing by rotation</td>
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Conclusion

We have

Weak Dhar Criterion efficient for biperiodic configurations
▶ Bijection with NCRSFs, extending the definition of biperiodic recurrent configurations
▶ Invariant distribution of external activity on NCRSFs and order given by a direction
▶ Involution on NCRSFs for atomic rotation preserving this distribution
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- What about other orders?
  - Experiments: periodic decreasing orders towards a direction is enough:
    $$e \prec_E f \Rightarrow e + (iW, jH) \prec_E f + (iW, jH) \text{ and } \langle s, (iW, jH) \rangle > 0 \Rightarrow e + (iW, jH) \prec_E e$$
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  - Only decreasing, or only periodic
  - Anything else
THANK YOU
Markov Chain for $G = (V \cup \{S\}, E)$

- States: stable configurations on $G$
- Transition: Add a particle to a vertex chosen uniformly and stabilize

The recurrent states are called recurrent configurations.

The stationary distribution is uniform on the recurrent configurations.

**Dhar Criterion** A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)
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Figure: Each non blue zone is described by a quadratic form. [arxiv:1708.09432]
Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

\[ M(a, b, c) = \begin{pmatrix} c + a & b \\ b & c - a \end{pmatrix} \]

The number of topples is:

\[ h(x) = \left[ \frac{1}{2} x^t M(a, b, c) x \right] \]

\[ = (c + a)x^2 + 2bxy + (c - a)y^2 \]

Sample for \( M(0.25, 0.875, 2.125) \)
Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

\[ M(a, b, c) = \begin{pmatrix} c + a & b \\ b & c - a \end{pmatrix} \]

The number of topples is:

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Then number of grains is

\[ \Delta h(u) = \sum_{v \sim u} h(v) - h(u). \]
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- It's periodic for \( a, b, c \in \mathbb{Q} \)
- But it may be negative and/or unstable!

It may be stabilized without changing density of grains.
A definition of recurrence for periodic stable configurations

Pattern + two dimensional period \((\vec{p}_1, \vec{p}_2)\).

\[ \forall x \in \mathbb{Z}^2 \quad u(x) = u(x + \vec{p}_1) = u(x + \vec{p}_2) \]
Lemme

If a periodic configuration is recurrent, then there exists a position \( y = t_1 \) for which all vertices of the first period are toppled.

We have \( \text{Period} 1 \subset E_{0,t_1} \Rightarrow E_{0,t_1} = E_{H,t_1-H} \) and \( v \in E_{0,t_1} \Rightarrow v + H \vec{y} \in E_{H,t_1} \). Then \( E_{H,t_1} \supset \text{Period} 2 \).