Two first-order logics of permutations

Valentin Féray
(joint work with Michael Albert and Mathilde Bouvel)

Institut für Mathematik, Universität Zürich

Journées de Combinatoire de Bordeaux, Février 2019
What is a permutation?

Answer 1: a bijection from $f_1; \ldots; n$ to $f_1; \ldots; n$.

Natural point of view to study the group structure, the cycle decomposition, ... Answer 2: a collection of dots in a square grid, with exactly one dot per row or per column.

Natural point of view to study patterns.

General belief: these two points of view are orthogonal.

We will give a mathematical statement supporting this belief using formal logic.
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We will give a mathematical statement supporting this belief using **formal logic**.
We will define two first-order logical theories:

- one representing \textit{permutations-as-bijections};
- the other representing \textit{permutations-as-diagrams}.

**Expressibility:** For each theory, some properties of permutations are expressible (i.e. there exist a first-order formula describing the property), some are not.

**Theorem (Albert, Bouvel & F. '18)**

Let \( P \) be a property expressible in both logical theories. Then \( P \) is in some sense trivial.
Informal presentation of the main result

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- one representing \textit{permutations-as-bijections};
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**Theorem (Albert, Bouvel & F. ’18)**

\textit{Let $P$ be a property expressible in both logical theories. Then $P$ is in some sense trivial.}
Models: pair \((A, R)\), where \(A\) is a (finite) set and \(R\) is a binary relation on \(A\).

Any permutation \(\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}\) is a model of the theory.

\[
A_\sigma := \{1, \ldots, n\}
\]

\[
\forall x R_\sigma y \iff y = \sigma(x)
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Conversely, any model is isomorphic to some $(A_\sigma, R_\sigma)$. 
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Conversely, any model is isomorphic to some \((A_\sigma, R_\sigma)\).

**Remark:** Models corresponding to conjugate permutations are isomorphic!
First-order formulas: every formula that you can write with quantifiers $\exists$, $\forall$, conjunctions $\land$, disjunctions $\lor$, negation $\neg$ and the relation $R$.

Examples:

- $\sigma$ has a fixed point: $\exists x, \ xRx$.
- $\sigma$ is an involution: $\forall x, \exists y, \ xRy \land yRx$.

Important: variables represent elements of the permutation, not sets!
TOTO: Theory Of Two Orders (models)

Models: pair \((A, <_P, <_V)\), where \(A\) is a (finite) set and \(<_P, <_V\) are binary relations on \(A\).

Any permutation (seen as a diagram) is a model of the theory.

\[ A_\sigma := \{ \text{dots in the diagram} \}; \]

\((x <_P y) \iff x \text{ is on the left of } y; \]

\((x <_V y) \iff x \text{ is below } y; \]

\[ A <_P B <_P C <_P D <_P E; \]

\[ B <_V E <_V A <_V C <_V D. \]
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Models: pair \((A, <_P, <_V)\), where \(A\) is a (finite) set and \(<_P, <_V\) are binary relations on \(A\).

Axioms: \(<_P\) and \(<_V\) are total orders.

Any permutation (seen as a diagram) is a model of the theory.

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A_\sigma := \{\text{dots in the diagram}\};
\]

\[
(x <_P y) \iff x \text{ is on the left of } y;
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\]

Conversely, any model is isomorphic to a permutation (and to exactly one).

\[
A <_P B <_P C <_P D <_P E;
\]

\[
B <_V E <_V A <_V C <_V D.
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TOTO: Theory Of Two Orders (formulas)

First-order formulas: every formula that you can write with quantifiers $\exists$, $\forall$, conjunctions $\land$, disjunctions $\lor$, negation $\neg$ and the relations $<_P$ and $<_V$.

Examples:

- $\sigma$ contains the pattern 213:
  $$\exists x, y, z : (x <_P y <_P z) \land (y <_V x <_V z).$$

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  $$\exists x, y, z : (x <_P y <_P z) \land (y <_V x <_V z) \land [\forall t, \neg(x <_P t <_P y)]$$

- Containment of all notions of generalized patterns, being a simple permutation.
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- Immediate: “σ contains the pattern 213” is not expressible in TOOB, since this property distinguishes some conjugate permutations.
So far, we have defined the logical theories and see examples of properties that they can describe (by giving an explicit formula).

Can we give examples of properties that they cannot describe?

**Immediate:** “\( \sigma \) contains the pattern 213” is not expressible in TOOB, since this property disguishes some conjugate permutations.

To go further, we need Ehrenfeucht-Fraïssé games, which we explain in the context of TOTO in the next few slides.
Ehrenfeucht-Fraïssé game \( \text{EF}(\pi, \pi', \ell) \)

**Data:** two permutations \( \pi, \pi' \) (the board of the game), an integer \( \ell \geq 1 \) (number of rounds).

**Two players:** Duplicator and Spoiler.
Ehrenfeucht-Fraïssé game $\text{EF}(\pi, \pi', \ell)$

**Data:** two permutations $\pi, \pi'$ (the board of the game), an integer $\ell \geq 1$ (number of rounds).

**Two players:** Duplicator and Spoiler.

**Each round:** Spoiler chooses an element in either $\pi$ or $\pi'$. Duplicator then chooses an element in the other permutation.

$\rightarrow$ denote $a_i$ and $a'_i$ the chosen elements in $\pi$ and $\pi'$.

**Note:** $a_i$ can have been chosen either by Spoiler or Duplicator.
**Ehrenfeucht-Fraïssé game** $\text{EF}(\pi, \pi', \ell)$

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**Who wins?** Duplicator wins if the $(a_1, \ldots, a_\ell)$ and $(a'_1, \ldots, a'_\ell)$ are in the same $P$ and $V$ orders.
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**Theorem (Ehrenfeucht-Fraïssé)**

*Duplicator has a winning strategy in the game $\text{EF}(\pi, \pi', \ell)$ if and only if $\pi$ and $\pi'$ satisfies the same TOTO formulas with quantifier depth at most $\ell$.***

**Notation:** $\pi \sim_\ell \pi'$. 
Ehrenfeucht-Fraïssé game (an example)

Note: $\pi$ contains a 213 pattern (in red), but not $\pi'$.

Reminder: there is a FO-formula of quantifier-depth 4, expressing "$\sigma$ contains a 213 pattern".

$\Rightarrow$ Spoiler should win the game $\text{EF}(\pi, \pi', 4)$.

Let us see her winning strategy.
In the first three rounds, **Spoiler** selects the 213 pattern in $\pi$, independently of what does **Duplicator**.
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**Duplicator** has to choose a 213 pattern in $\pi'$; since $\pi'$ has no 213, the dots $a'_1$ and $a'_2$ cannot be consecutive (in position).
In the first three rounds, **Spoiler** selects the $213$ pattern in $\pi$, independently of what does **Duplicator**.

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**Spoiler** chooses a point $a'_4$ between $a'_1$ and $a'_2$. **Duplicator** should choose a point $a_4$ between $a_1$ and $a_2$, but there is none.

$\implies$ **Spoiler** wins.
Ehrenfeucht-Fraïssé game (second example)

Let $\pi = \delta_{2^\ell - 1}$ and $\pi' = \delta_{2^\ell}$ are decreasing permutations of sizes $2^\ell - 1$ and $2^\ell$.

**Duplicator** has a winning strategy:

- if Spoiler plays near a corner or an already chosen dot, play at the same distance;
- if Spoiler plays far from corners/other dots, do the same.

For $\ell = 3$:
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![Diagram](image)
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![Diagram showing the game with dots placed at strategic positions to illustrate the winning strategy for Duplicator.](image)
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For $\ell = 3$:

![Diagram showing the game setup for $\ell = 3$.]
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For $\ell = 3$: 

```latex
\begin{align*}
  &a_1 & a_2 & a_3 \\
\end{align*}
```

```latex
\begin{align*}
  &a_1' & a_2' & a_3' \\
\end{align*}
```
A non-expressivity result

Proposition

There is no FO-formula in TOTO expressing the property “σ has a fixed point”.
A non-expressivity result

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*There is no FO-formula in TOTO expressing the property “σ has a fixed point”.*

Proof: Assume there is such a formula \( \Psi \) and denote \( \ell \) its quantifier depth.

\[\text{Since Duplicator wins the game EF (}2^{\ell_1}; 2^{\ell_2}\text{), either both} \quad 2^{\ell_1}\text{and} \quad 2^{\ell_2}\text{, or none of them does.}
\]

But \( 2^{\ell_1} \) has a fixed point, while \( 2^{\ell_2} \) does not.
A non-expressivity result

Proposition

There is no FO-formula in TOTO expressing the property “σ has a fixed point”.

Proof: Assume there is such a formula Ψ and denote ℓ its quantifier depth.

Since Duplicator wins the game EF(δ_{2ℓ−1}, δ_{2ℓ}), either both δ_{2ℓ−1} and δ_{2ℓ} satisfy Ψ, or none of them does.
A non-expressivity result

Proposition

There is no FO-formula in TOTO expressing the property “σ has a fixed point”.

Proof: Assume there is such a formula \( \Psi \) and denote \( \ell \) its quantifier depth.

Since Duplicator wins the game \( \text{EF}(\delta_{2^{\ell-1}}, \delta_{2^\ell}) \), either both \( \delta_{2^{\ell-1}} \) and \( \delta_{2^\ell} \) satisfy \( \Psi \), or none of them does.

But \( \delta_{2^{\ell-1}} \) has a fixed point, while \( \delta_{2^\ell} \) does not. \( \square \)
Reminder of expressibility of some properties:

**Having a fixed point** expressible in TOOB but not in TOTO;

**Containing a 231-pattern** expressible in TOTO but not in TOOB.

**Question**

What are the properties expressible in both TOTO and TOOB?

We expect not to find many, since, intuitively, the two logics consider permutations with orthogonal point of views...
Easy examples of properties expressible in both logics:

- Being in some conjugacy class $C_\lambda$ or in a finite union of those (or the negation);
Some properties expressible in TOTO and TOOB (1/3)

Easy examples of properties expressible in both logics:

- Being in some conjugacy class $C_{\lambda}$ or in a finite union of those (or the negation);
- Being an identity permutation of any size (also called increasing permutations).

TOOB  Each element is sent to itself ($\forall x, xRx$);

TOTO  Values and positions order coincide

\[(\forall x, y, (x <_V y) \Leftrightarrow (x <_P y)).\]
“Being a transposition” is also expressible in both logics!

- In TOOB, it’s trivial:
  
  There exist two points which are sent one to the other and all other points are fixed.
“Being a transposition” is also expressible in both logics!

- In TOOB, it’s trivial:
  
  \[
  \text{There exist two points which are sent one to the other and all other points are fixed.}
  \]

- To express it in TOTO, we observe that the diagram of a transposition has a very particular form.

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

\[
\text{There exist two points } x \text{ and } y \text{ which form an inversion and, for all other points, value and position order coincide, and the following holds: } z \text{ is smaller (resp. bigger) than both } x \text{ and } y \text{ in position order iff it also is in value order.}
\]
Set $D_\lambda = \bigcup_k C_{\lambda \cup (1^k)}$, i.e. the set of permutations whose non-fixed points form a permutation of type $\lambda$.

**Lemma**

*Being in $D_\lambda$ is expressible in both logics.*

Easy generalization of the expressibility of “being a transposition” (which corresponds to $\lambda = (2)$.)
Main theorem – weak form (Albert, Bouvel & F. 18)

Let \((P)\) be a property expressible in both TOOB and TOTO. Then
- either all permutations with sufficiently large support verify \((P)\),
- or there is a bound on the size of the support of permutations verifying \((P)\).
Main theorem(s)

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- either all permutations with sufficiently large support verify \((P)\),
- or there is a bound on the size of the support of permutations verifying \((P)\).

Main theorem – strong form (Albert, Bouvel & F. 18)
Consider the boolean algebra
\[ \mathcal{A} = \langle C_\lambda, D_\lambda, \lambda \text{ partition} \rangle, \]
i.e. the smaller collection of sets containing all \(C_\lambda\)'s, all \(D_\lambda\)'s, and closed by taking unions, intersections and complements.
Then being in some set \(A\) is expressible in both TOOB and TOTO if and only if \(A\) is in \(\mathcal{A}\).
Let \((P)\) be a property expressible in both TOOB and TOTO, and let \(\ell\) be the quantifier depth of its TOTO FO formula.

**Lemma (changing length of long full cycles)**

*If \((P)\) is satisfied by some full cycles of length \(n_1 \geq 2^\ell\), then all full cycles of length at least \(2^\ell\) satisfy \((P)\).*
Strategy of proof (1/4)

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Lemma (changing length of long full cycles)

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**Proof:** Since \((P)\) is expressible in TOOB, it is invariant by conjugacy. Thus the following full cycle satisfies \((P)\):

![Diagram of a full cycle](image)

size \(n_1\)
Strategy of proof (1/4)

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\[
\begin{array}{c}
\text{size } n_1 \\
\bullet \quad \bullet
\end{array} \quad \sim_{\ell} \quad \begin{array}{c}
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\bullet \quad \bullet
\end{array}
\]

whenever \(n_2 \geq 2^\ell\).
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Lemma (changing length of long full cycles)

If $(P)$ is satisfied by some full cycles of length $n_1 \geq 2^{\ell}$, then all full cycles of length at least $2^{\ell}$ satisfy $(P)$.

Proof: Since $(P)$ is expressible in TOOB, it is invariant by conjugacy. Thus the following full cycle satisfies $(P)$:

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whenever $n_2 \geq 2^{\ell}$. Thus the full-cycle on the RHS satisfies $(P)$. Using again invariance of $(P)$ by conjugacy, all full cycles of all sizes $n_2 \geq 2^{\ell}$ satisfy $(P)$. 

Strategy of proof (2/4)

Another lemma of the same flavour. Take $k \geq 2^\ell - 1$.

**Lemma (Changing repeated short cycles into a long cycle)**

$(P)$ is satisfied by some/all permutations of type $(2, \ldots, 2)$ ($k$ times), iff it is satisfied by some/all permutations of type $(2k + 1)$. 
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**Proof:**

Segments of the same size type $(2, \ldots, 2)$.

Sizes of the segment differ by 1 type $(2k + 1)$.

(Easy to adapt to $(m, \ldots, m)$ instead of $(2, \ldots, 2)$ for $m \geq 2$.)
Lemma (absorbing a small cycle)

\( (P) \) is satisfied by some/all full cycles of length \( n \geq 2^\ell \), iff some/all permutations of type \( (n-2, 1) \) also satisfy \( (P) \).

Proof (for \( n \equiv 2 \pmod{4} \)):

sizes of the segment differ by 1

type \((n-2, 1)\).

\( \sim_\ell \)

two segments of the same size

type \( (n) \);

(Easy to adapt to \((n-k-1, k)\) instead of \((n-2, 2)\).)
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Then (third lemma), we can absorb other cycles, so that $(P)$ is satisfied by some/all full cycles.
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Go in the other direction: \((P)\) is satisfied by all permutations with large support.
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Going from the weak to the strong form is relatively easy.
Other results (1/2)

Stack sorting operator $S$ defined by: $S(LnR) = S(L)S(R)n$.

Knuth (’68) $S(\sigma) = \text{id}$ iff $\sigma \in \text{Av}(231)$;

West (’93) characterization of $\sigma$’s s.t. $S^2(\sigma) = \text{id}$;

Ulfarsson (’12) characterization of $\sigma$’s s.t. $S^3(\sigma) = \text{id}$. 
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Ulfarsson (’12) characterization of $\sigma$’s s.t. $S^3(\sigma) = \text{id}$.

Theorem (Albert, Bouvel & F. ’18)

For all $\ell \geq 1$, there is a constructible TOTO formula expressing the property $S^\ell(\sigma) = \text{id}$. 
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For \( \sigma \in \mathcal{C} = \text{Av}(321) \), we have:

\[
\text{there is a fixed point} \quad \iff \quad \text{there is a dot } x \text{ with no dots at its top left, nor at its bottom right.}
\]
Other results (2/2)

TOTO(\(\mathcal{C}\)): same theory as TOTO except that we add axioms to keep only permutations in a given class \(\mathcal{C}\).

For \(\sigma \in \mathcal{C} = \text{Av}(321)\), we have:

- there is a fixed point \iff there is a dot \(x\) with no dots at its top left, nor at its bottom right.

\[ \rightarrow \text{“having a fixed point” is expressible in TOTO(\text{Av}(321)).} \]
TOTO(\mathcal{C})$: same theory as TOTO except that we add axioms to keep only permutations in a given class \( \mathcal{C} \).

For \( \sigma \in \mathcal{C} = \text{Av}(321) \), we have:

there is a fixed point \iff there is a dot \( x \) with no dots at its top left, nor at its bottom right.

\[ \rightarrow \text{“having a fixed point” is expressible in TOTO(\text{Av}(321)).} \]

Theorem (Albert, Bouvel & F. ’18)

“Having a fixed point” is not expressible in TOTO(\mathcal{C}) if and only if \( \mathcal{C} \) contains either all decreasing permutations or all permutations of the form \( + \).

+ some extension (but no complete characterization) for longer cycles.
Logic and random permutations

Context: many beautiful results on 0-1/convergence laws for random graphs (see Marc’s talk). What about permutations?

Conjecture (Albert, Bouvel & F. ’18)

There is a convergence law for TOTO for uniform random permutations. Namely, let $n$ be a uniform random permutation of size $n$. Then for any TOTO FO-formula $(P)$, $P(n)$ satisfies $(P)$ has a limit as $n \to \infty$.

Example: $P(n)$ has an adjacency*

*Adjacency: two consecutive entries with consecutive values (in any order).
Logic and random permutations

Context: many beautiful results on 0-1/convergence laws for random graphs (see Marc’s talk). What about permutations?

Conjecture (Albert, Bouvel & F. ’18)
There is a convergence law for TOTO for uniform random permutations.

Namely, let $\sigma_n$ be a uniform random permutation of size $n$. Then for any TOTO FO-formula $(P)$,

$$\mathbb{P}(\sigma_n \text{ satisfies } (P))$$

has a limit as $n \to \infty$.

Example: $\mathbb{P}(\sigma_n \text{ has an adjacency}^*) \to \frac{1}{e^2}$.

*Adjacency: two consecutive entries with consecutive values (in any order).
Thank you for your attention!