Marches dans trois quarts de plan

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How can we enumerate walks in the three quarter plan?

I. Partition the walks by their endpoints

II. Transform the cone

III. Integral expression of the generating function
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1. Introduction to walks models

2. A first functional equation

3. Functional equations for the $3\pi/4$-cone

4. Transforming the cones

5. Analytic method in the case of a symmetric step set

6. Integral expression of the generating function

7. Further objectives and perspectives
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Introduction to walk models

A walk is a sequence of jumps from a step set \( S \).

We are interested in walks that start at the origin, \textbf{take steps in} \( S \) and \textbf{remain in the three quarter plane}.

\[
S = \{(1,0), (0,1), (-1,0), (0,-1)\}
\]
Introduction to walk models

A walk is a sequence of jumps from a step set $S$.

We are interested in walks that start at the origin, **take steps in** $S$ and remain in the **three quarter plane**.

$$S = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$$

**Objective**

The goal is to **compute** the **number of paths** $c_{i,j}(n)$ of length $n$, starting at $(0, 0)$ and ending at $(i, j)$, with $(i \geq 0 \text{ or } j \geq 0)$ and $n \geq 0$. 

Walks in the three quarter plane

We focus on walks with small steps: \( S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\} \)

We study the generating function of walks with small steps

\[
C(x, y) = \sum_{n \geq 0} \sum_{(i,j) \in \mathcal{C}} c_{i,j}(n) x^i y^j t^n
\]

Strategy

We want to find an explicit expression of the generating function \( C(x, y) \).
Context

M. Bousquet-Mélou, 2016. *Square lattice walks avoiding a quadrant.*
- Simple and Diagonal walks with various starting points
- Exact expression of the generating function
- Nature of the generating function
- New proof of Gessel’s conjecture via reflection principle

- Computation of the asymptotic of the number of excursions

- Enumerating formulas for planar walks keeping track of the winding angle
- Can be used to enumerate simple walks in the three quarter plane
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A combinatorial decomposition - $S = \{\pm 1, 0\}^2 \setminus \{(0, 0)\}$
A first functional Equation in the three quarter plane

The combinatorial decomposition gives a generating function equation

\[ C(x, y) = 1 + tS(x, y)C(x, y) - t \left( \sum_{(i,-1) \in S} x^i y^{-1} \right) C_0(x^{-1}) \]

\[- t \left( \sum_{(-1,j) \in S} x^{-1} y^j \right) C_0(y^{-1}) + t \delta_{-1,-1} C(0,0). \]

Where

\[ C(x, y) = \sum_{i \geq 0 \text{ or } j \geq 0} c_{i,j}(n) x^i y^j t^n, \quad C_0(x^{-1}) = \sum_{i \leq 0, n \geq 0} c_{i,0}(n) x^i t^n, \]

\[ C_0(y^{-1}) = \sum_{j \leq 0, n \geq 0} c_{0,j}(n) y^j t^n, \quad C_{0,0} = \sum_{n \geq 0} c_{0,0}(n) t^n. \]
A first functional equation

Compare with the functional equation in the quarter plane

\[ Q(x, y) = 1 + tS(x, y)Q(x, y) - t \left( \sum_{(i,-1) \in S} x^i y^{-1} \right) Q(x, 0) \]

\[ - t \left( \sum_{(-1,j) \in S} x^{-1} y^j \right) Q(0, y) + t \delta_{-1,-1} Q(0, 0). \]

Where

\[ Q(x, y) = \sum_{i \geq 0, j \geq 0} q_{i,j}(n) x^i y^j t^n, \quad Q(x, 0) = \sum_{i \geq 0, n \geq 0} q_{i,0}(n) x^i t^n, \]

\[ Q(0, y) = \sum_{j \geq 0, n \geq 0} q_{0,j}(n) y^j t^n, \quad Q(0, 0) = \sum_{n \geq 0} q_{0,0}(n) t^n. \]
A first functional equation in the three quarter plane

\[ C(x, y) = 1 + tS(x, y)C(x, y) - t \left( \sum_{(i,-1) \in S} x^i y^{-1} \right) C_0(x^{-1}) \]

\[ - t \left( \sum_{(-1,j) \in S} x^{-1} y^j \right) C_0(y^{-1}) + t \delta_{-1,-1} C(0, 0). \]

Where

\[ C(x, y) = \sum_{i \geq 0 \text{ or } j \geq 0} c_{i,j}(n) x^i y^j t^n, \quad C_0(x^{-1}) = \sum_{i \leq 0, n \geq 0} c_{i,0}(n) x^i t^n, \]

\[ C_0(y^{-1}) = \sum_{j \leq 0, n \geq 0} c_{0,j}(n) y^j t^n, \quad C_{0,0} = \sum_{n \geq 0} c_{0,0}(n) t^n. \]
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Strategy

Cut the domain into three parts

We decompose the domain of possible ends of the walks into three parts:

\[ C(x, y) = L(x, y) + D(x, y) + U(x, y). \]

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Aside: Join-the-Shortest-Queue model (JSQ)

We consider a model with two queues in which the customers choose the shortest queue (if the two queues have same length, then the customers choose a queue according to a fixed probability law).
\[ S = \{ \pm 1, 0 \}^2 \setminus \{(0, 0), (-1, 1), (1, -1)\} \]
\[ \mathcal{S} = \{ \pm 1, 0 \}^2 \setminus \{ (0, 0), (-1, 1), (1, -1) \} \]

\[
L(x, y) = tS(x, y)L(x, y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x, y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^\ell(x, y) \\
- t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_0(y^{-1}) + t\delta_{-1,0}x^{-1} \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n
\]
$\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0, 0), (-1, 1), (1, -1)\}$

\[
L(x, y) = tS(x, y)L(x, y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x, y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^\ell(x, y) \\
- t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_0-(y^{-1}) + t\delta_{-1,0}x^{-1}\sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n
\]

\[
U(x, y) = tS(x, y)U(x, y) + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D(x, y) - t(\delta_{0,-1}y^{-1} + \delta_{1,0}x)D^u(x, y) \\
- t(\delta_{0,-1}y^{-1} + \delta_{-1,-1}x^{-1}y^{-1})U_0(x^{-1}) + t\delta_{0,-1}y^{-1}\sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n
\]
\[ S = \{ \pm 1, 0 \}^2 \setminus \{(0, 0), (-1, 1), (1, -1)\} \]

\[
L(x, y) = tS(x, y)L(x, y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x, y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^\ell (x, y) \\
- t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_0(y^{-1}) + t\delta_{-1,0}x^{-1} \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n
\]

\[
U(x, y) = tS(x, y)U(x, y) + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D(x, y) - t(\delta_{0,-1}y^{-1} + \delta_{1,0}x)D^u (x, y) \\
- t(\delta_{0,-1}y^{-1} + \delta_{-1,-1}x^{-1}y^{-1})U_0(x^{-1}) + t\delta_{0,-1}y^{-1} \sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n
\]

\[
D(x, y) = 1 + t(\delta_{1,1}xy + \delta_{-1,-1}x^{-1}y^{-1})D(x, y) - t\delta_{-1,-1}x^{-1}y^{-1}c_{0,0}(n) \\
+ t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D^u (x, y) - t\delta_{0,-1}y^{-1} \sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n \\
+ t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D^\ell (x, y) - t\delta_{-1,0}x^{-1} \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n
\]
\[ S = \{ \pm 1, 0 \}^2 \setminus \{(0,0), (-1,1), (1,-1)\} \]

\[
L(x,y) = tS(x,y)L(x,y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x,y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^{\ell}(x,y) - t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_0(y^{-1}) + t\delta_{-1,0}x^{-1} \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n
\]

\[
U(x,y) = tS(x,y)U(x,y) + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D(x,y) - t(\delta_{0,-1}y^{-1} + \delta_{-1,-1}x^{-1}y^{-1})U_0(x^{-1}) + t\delta_{0,-1}y^{-1} \sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n
\]

\[
D(x,y) = 1 + t(\delta_{1,1}xy + \delta_{-1,-1}x^{-1}y^{-1})D(x,y) - t\delta_{-1,-1}x^{-1}y^{-1}c_{0,0}(n) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D^u(x,y) - t\delta_{0,-1}y^{-1} \sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D^{\ell}(x,y) - t\delta_{-1,0}x^{-1} \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n
\]
For any step set with no jumps \((-1, 1)\) or \((1, -1)\) and start at \((0, 0)\), one has:

\[ L(x, y)K(x, y) = -xy - \left[ t \left( \delta_{-1,-1} + \delta_{0,1}x^2y + \delta_{0,-1}x + \delta_{1,1}x^2y^2 \right) - xy \right] D(x, y) \]

\[ + t\left( \delta_{-1,-1} + \delta_{-1,0}y \right) L_0(y) - t \left( \delta_{0,-1}x + \delta_{1,0}x^2y \right) D^u(x, y) \]

\[ + t\delta_{0,-1}x \sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n + t\delta_{-1,-1}c_{0,0}(n). \]

\[ U(x, y)K(x, y) = -xy - \left[ t \left( \delta_{-1,-1} + \delta_{0,1}xy^2 + \delta_{-1,0}y + \delta_{1,1}x^2y^2 \right) - xy \right] D(x, y) \]

\[ + t\left( \delta_{-1,-1} + \delta_{0,-1}x \right) U_0(x) - t \left( \delta_{-1,0}y + \delta_{0,1}xy^2 \right) D^\ell(x, y) \]

\[ + t\delta_{-1,0}y \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n + t\delta_{-1,-1}c_{0,0}(n). \]
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Transforming the cones

\[ L(x, y) \]

\[ U(x, y) \]

\[ D(x, y) \]
Transforming the cones

$$U(x, y)$$

$$L(x, y)$$

$$D(x, y)$$

$$\varphi_U$$

$$\varphi_L$$

$$U_{\varphi_U}(x, y)$$

$$L_{\varphi_L}(x, y)$$

$$D_{\varphi_L}(y) = D_{\varphi_U}(y)$$
Transforming the cones

$D(x, y) = U\varphi_U(x, y) = L\varphi_L(x, y)$

Symmetric case

$Q(x, y)$
### Change of variables

<table>
<thead>
<tr>
<th>( \varphi_U(S) )</th>
<th>( S )</th>
<th>( \varphi_L(S) )</th>
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</thead>
<tbody>
<tr>
<td>Gouyou-Beauchamps</td>
<td>Simple Walks</td>
<td>Gessel</td>
</tr>
<tr>
<td>Big step Walks</td>
<td>Diagonal Walks</td>
<td>Big steps Walks</td>
</tr>
</tbody>
</table>

\[
\varphi_L(x, y) = (xy, x^{-1})
\]

\[
\varphi_U(x, y) = (x, x^{-1}y)
\]

\[
\begin{cases}
U_{\varphi_U}(x, y) = U(\varphi_U(x, y)); \\
D_{\varphi_U}(y) = D(\varphi_U(x, y)); \\
K_{\varphi_U}(x, y) = K(\varphi_U(x, y)).
\end{cases}
\]

\[
\begin{cases}
L_{\varphi_L}(x, y) = L(\varphi_L(x, y)); \\
D_{\varphi_L}(y) = D(\varphi_L(x, y)); \\
K_{\varphi_L}(x, y) = K(\varphi_L(x, y)).
\end{cases}
\]
New system of functional equations

For any step set with no jumps \((-1, 1)\) or \((1, -1)\) and start at \((0, 0)\), one has:

\[
L_{\varphi_L}(x, y)K_{\varphi_L}(x, y) = -xy - \left[ t \left( \delta_{-1,-1}x + \delta_{1,0}x^2y^2 + \delta_{0,-1}x^2y + \delta_{1,1}xy^2 \right) - xy \right] D_{\varphi_L}(y) \\
+ t \left( \delta_{-1,-1}x + \delta_{-1,0} \right) L_{\varphi_L}(x, 0) - t \left( \delta_{0,-1}x + \delta_{0,1}xy \right) D^{U}_L(y) \\
+ t\delta_{0,-1}x^2y \sum_{n \geq 0} c_{-1,0}(n)x^{-1}t^n + t\delta_{-1,-1}xc_{0,0}(n).
\]

\[
U_{\varphi_U}(x, y)K_{\varphi_U}(x, y) = -xy - \left[ t \left( \delta_{-1,-1}x + \delta_{0,1}y^2 + \delta_{-1,0}y + \delta_{1,1}xy^2 \right) - xy \right] D_{\varphi_U}(x, y) \\
+ t \left( \delta_{-1,-1}x + \delta_{0,-1}x^2 \right) U_{\varphi_U}(x) - t \left( \delta_{-1,0}x + \delta_{0,1}xy \right) D^{L}_{\varphi_L}(y) \\
+ t\delta_{-1,0}y \sum_{n \geq 0} c_{0,-1}(n)y^{-1}t^n + t\delta_{-1,-1}xc_{0,0}(n).
\]
A symmetric example
A symmetric example
A symmetric example

Different approach

Beaton, Owczarek and Rechnitzer, 2018. *Exact solution of some quarter plane walks with interacting boundaries.*
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A single functional Equation

\[ K_{\varphi_L}(x, y)L_{\varphi_L}(x, y) = c_{\varphi_L}(x)L_{\varphi_L}(x, 0) - x \left[ x\tilde{a}_{\varphi_L}(y) + \frac{1}{2}\tilde{b}_{\varphi_L}(y) \right] D_{\varphi_L}(y) \\
- \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0). \]
The Kernel of the Walks

\[ K(x, y) = xy [tS(x, y) - 1]. \]

The Kernel: a polynomial of degree 2 in \(x\) and in \(y\)

\[ K(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) = a(x)y^2 + b(x)y + c(x). \]
The Kernel of the Walks

\[ K(x, y) = xy [tS(x, y) - 1]. \]

The Kernel: a polynomial of degree 2 in \( x \) and in \( y \)

\[ K(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) = a(x)y^2 + b(x)y + c(x). \]

Discriminant: \( \tilde{d}(y) = \tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y) \) and \( d(x) = b(x)^2 - 4a(x)c(x). \)
The Kernel of the Walks

\[ K(x, y) = xy [tS(x, y) - 1]. \]

The Kernel: a polynomial of degree 2 in \( x \) and in \( y \)

\[ K(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) = a(x)y^2 + b(x)y + c(x). \]

Discriminant: \( \tilde{d}(y) = \tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y) \) and \( d(x) = b(x)^2 - 4a(x)c(x) \).

Zeros of the Kernel, \( i = 0, 1 \):

\[ K(X_i(y), y) = 0 \quad \text{and} \quad K(x, Y_i(x)) = 0 \]

\[ X_i(y) = \frac{-\tilde{b}(y) \pm \sqrt{\tilde{d}(y)}}{2\tilde{a}(y)} \quad \text{and} \quad Y_i(x) = \frac{-b(x) \pm \sqrt{d(x)}}{2a(x)}. \]

The roots of the kernel define analytic curves.
Branches of the Kernel

Kernel for Gessel’s Walks

\[ K(x, y) = xy \left[ t \left( x + xy + x^{-1} + x^{-1}y^{-1} \right) - 1 \right]. \]

\( X([y_1, y_2]) \) for Gessel’s walk \( (t = 1/8) \).

\( Y([x_1, x_2]) \) for Gessel’s walk \( (t = 1/8) \).
Branches of the Kernel

Kernel for Simple Walks

\[ K(x, y) = xy \left[ t (x + y + x^{-1} + y^{-1}) - 1 \right]. \]

\[ X([y_1, y_2]) \text{ for Simple walk.} \]

\[ Y([x_1, x_2]) \text{ for Simple walk.} \]
Generating function $D(y)$ stated as a BVP

Functional Equation

\[
K_{\varphi_L}(x, y)L_{\varphi_L}(x, y) = c_{\varphi_L}(x)L_{\varphi_L}(x, 0) - x \left[ x\tilde{a}_{\varphi_L}(y) + \frac{1}{2}\tilde{b}_{\varphi_L}(y) \right] D_{\varphi_L}(y)
- \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).
\]
Generating function $D(y)$ stated as a BVP

**Functional Equation**

\[
K_{\varphi_L}(x, y)L_{\varphi_L}(x, y) = c_{\varphi_L}(x)L_{\varphi_L}(x, 0) - x \left[ x\bar{a}_{\varphi_L}(y) + \frac{1}{2}\bar{b}_{\varphi_L}(y) \right] D_{\varphi_L}(y)
- \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).
\]

**Riemann-Carleman with shift BVP**

By evaluating the functional equation in $Y_{\varphi_L,0}$ and $Y_{\varphi_L,1}$, we can transform the functional equation into the following **boundary value problem**:

For $y \in Y_{\varphi_L}([x_{\varphi_L,1}, x_{\varphi_L,2}])$,

\[
\sqrt{d_{\varphi_L}(y)}D_{\varphi_L}(y) - \sqrt{d_{\varphi_L}(\bar{y})}D_{\varphi_L}(\bar{y}) = -y + \bar{y}.
\]
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Contour-integral expression for $D(y)$

Theorem [Raschel, T., 2018]

Consider a symmetric step set that does not contain the steps $(1, -1)$ and $(-1, 1)$. The diagonal section have the following contour-integral expression:

$$D_{\varphi_L}(y) = \frac{-w'_{\varphi_L}(y)}{\sqrt{(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L}, 1)))(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L}, 2)))}}$$

$$\frac{1}{2i\pi} \int_{\mathcal{L}_{\varphi_L}} \frac{zw'_{\varphi_L}(z)}{\sqrt{w_{\varphi_L}(z) - w_{\varphi_L}(y_{\varphi_L}, 1)(w_{\varphi_L}(z) - w_{\varphi_L}(y))}} \, dz.$$ 

With $w_{\varphi_L}$ a known conformal mapping, and $\mathcal{L}_{\varphi_L}$ the contour $Y_{\varphi_L}([x_{\varphi_L}, 1, x_{\varphi_L}, 2])$. 

Example of the Simple walk

$w_{\varphi_L}$ (algebraic) can be expressed in terms of Weierstrass elliptic function.

Integral expression for $D_{\varphi_L}(y)$

$$D_{\varphi_L}(y) = \frac{-w'_{\varphi_L}(y)}{\sqrt{(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L},1)))(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L},2)))}}$$

$$\frac{1}{2i\pi} \int_{L_{\varphi_L}} \frac{zw'_{\varphi_L}(z)}{\sqrt{w_{\varphi_L}(z) - w_{\varphi_L}(y_{\varphi_L,1})(w_{\varphi_L}(z) - w_{\varphi_L}(y))}} \, dz.$$
Summary

Remember - Functional Equation

\[ K_{\phi_L}(x, y) L_{\phi_L}(x, y) = c_{\phi_L}(x) L_{\phi_L}(x, 0) - x \left[ x \tilde{a}_{\phi_L}(y) + \frac{1}{2} \tilde{b}_{\phi_L}(y) \right] D_{\phi_L}(y) \]
\[ - \frac{1}{2} xy + \frac{1}{2} tx \delta_{-1,-1} D_{\phi_L}(0). \]

Remember - Domain in three parts

\[ C(x, y) = L(x, y) + D(x, y) + U(x, y). \]

Symmetry of the cut and the walk

\[ \Rightarrow U(x, y) = L(y, x). \]
Summary

Remember - Functional Equation

\[ K_{\varphi_L}(x, y)L_{\varphi_L}(x, y) = c_{\varphi_L}(x)L_{\varphi_L}(x, 0) - x \left[ x\tilde{a}_{\varphi_L}(y) + \frac{1}{2}\tilde{b}_{\varphi_L}(y) \right] D_{\varphi_L}(y) \]

\[ - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0). \]

Remember - Domain in three parts

\[ C(x, y) = L(x, y) + D(x, y) + L(y, x). \]

- We have an expression of \( D_{\varphi_L}(y); \)
Summary

Remember - Functional Equation

\[ K_{\varphi_L}(x, y)L_{\varphi_L}(x, y) = c_{\varphi_L}(x)L_{\varphi_L}(x, 0) - x \left[ x\tilde{a}_{\varphi_L}(y) + \frac{1}{2}\tilde{b}_{\varphi_L}(y) \right] D_{\varphi_L}(y) \]
\[ - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1, -1}D_{\varphi_L}(0). \]

Remember - Domain in three parts

\[ C(x, y) = L(x, y) + D(x, y) + L(y, x). \]

- We have an expression of $D_{\varphi_L}(y)$;
- With the functional equation we get an expression of $L_{\varphi_L}(x, y)$;
Summary

Remember - Functional Equation

\[ K_{\varphi_L}(x, y)L_{\varphi_L}(x, y) = c_{\varphi_L}(x)L_{\varphi_L}(x, 0) - x\left[x\tilde{a}_{\varphi_L}(y) + \frac{1}{2}\tilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0). \]

Remember - Domain in three parts

\[ C(x, y) = L(x, y) + D(x, y) + L(y, x). \]

- We have an expression of \( D_{\varphi_L}(y) \);
- With the functional equation we get an expression of \( L_{\varphi_L}(x, y) \);
- With a change of variable we get an expression of \( D(x, y) \) and \( L(x, y) \);
Summary

Remember - Functional Equation

\[ K_{\varphi L}(x, y)L_{\varphi L}(x, y) = c_{\varphi L}(x)L_{\varphi L}(x, 0) - x \left[ x\tilde{a}_{\varphi L}(y) + \frac{1}{2}\tilde{b}_{\varphi L}(y) \right] D_{\varphi L}(y) - \frac{1}{2} xy + \frac{1}{2} tx\delta_{-1,-1}D_{\varphi L}(0). \]

Remember - Domain in three parts

\[ C(x, y) = L(x, y) + D(x, y) + L(y, x). \]

- We have an expression of \( D_{\varphi L}(y) \);
- With the functional equation we get an expression of \( L_{\varphi L}(x, y) \);
- With a change of variable we get an expression of \( D(x, y) \) and \( L(x, y) \);
- Then we have an expression of \( C(x, y) \).
Content

1. Introduction to walks models
2. A first functional equation
3. Functional equations for the $3\pi/4$-cone
4. Transforming the cones
5. Analytic method in the case of a symmetric step set
6. Integral expression of the generating function
7. Further objectives and perspectives
Further Objectives and Perspectives

Limitations of our method

We made a restriction to:

- Symmetric step sets.
- Step sets with no jumps \((-1, 1)\) and \((1, -1)\).

Otherwise, we have to deal with too many variables and/or with walks with big steps.

Objectives

- Simplify the contour integral expression.
- Series expansion of the generating function.
References


